Concepts of Mathematics

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Lecture 6

1 Number Systems

Consider an infinite straight line; we mark the line into equal distance segments, with numbers $0, 1, 2, 3, \ldots$ and $-1, -2, -3, \ldots$, and so on. We think of every point on the line is a real number, and the line is called the **real line** or **real axis**, denoted $\mathbb{R}$. There is a natural ordering on $\mathbb{R}$: for two real numbers $x, y \in \mathbb{R}$, if x is on the left of y, we write $x < y$ or $y > x$. Also, $x \le y$ indicates that either $x < y$ or $x = y$.

The **integers** are the whole numbers marked on the line; the set of integers is denoted by $\mathbb{Z}$. Fraction $\frac{m}{n}$ with integers m, n can be marked on the line, they are called **rational numbers**. Real numbers which are not rational are called **irrational**. For rational numbers $\frac{a}{b}$, $\frac{c}{d}$, they can be added and multiplied as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Definition 1 (Addition and Multiplication of Real Numbers)**.** For real numbers $a, b, c \in \mathbb{R}$,

(1) $a + b = b + a, \quad ab = ba.$

(2) $a + (b + c) = (a + b) + c, \quad a(bc) = (ab)c.$

(3) $a(b + c) = ab + ac.$

(4) If $a \neq 0$, then there exists a unique real number $x \neq 0$ such that $ax = 1$; we write $x = a^{-1} = \frac{1}{a}$ and

$$\frac{b}{a} := a^{-1}b.$$

(5) If $a < b$, then $a + c < b + c$.

(4) If $a < b$ and $c > 0$, then $ac < bc$.

Proposition 2. *Between any two rational numbers there exists another rational number.*

Proof. Let r, s be two distinct rational numbers such that $r < s$. We write $r = \frac{a}{b}$ and $s = \frac{c}{d}$. Let $t = \frac{1}{2}(r + s)$. Clearly, $t = \frac{ad+bc}{2bd}$ is rational. Since $\frac{1}{2}s > \frac{1}{2}r$, then

$$t = \frac{1}{2}(r + s) = \frac{1}{2}r + \frac{1}{2}s > \frac{1}{2}r + \frac{1}{2}r = r,$$

$$t = \frac{1}{2}(r + s) = \frac{1}{2}r + \frac{1}{2}s < \frac{1}{2}s + \frac{1}{2}s = s.$$

□

Proposition 3. *$\sqrt{2}$ is irrational.*

Proof. Suppose $\sqrt{2}$ is rational, say $\sqrt{2} = \frac{m}{n}$, where m and n are integers having no common factors. Then $2 = \frac{m^2}{n^2}$, i.e., $m^2 = 2n^2$. Clearly, m^2 is even. So m must be even. Write $m = 2k$. Then $m^2 = 4k^2 = 2n^2$. It follows that $n^2 = 2k^2$ is even. By the same token, we see that n is even. Hence $\frac{m}{n}$ is not in reduced form. This is a contradiction. □

Proposition 4. *Let a be an irrational number and r a rational number. Then*

(1) *$a + r$ is irrational, and*

(2) *if $r \neq 0$, then ar is irrational.*

If a and b are distinct nonzero irrational real numbers then ab is irrational. (Wrong! Why?)

Proposition 5. *Between any two real numbers there is an irrational number.*

Proof. Let a and b be two real numbers with $a < b$. Choose a positive integer n such that $n > \frac{\sqrt{2}}{b-a}$.
Case 1: If a is rational, then $a + \frac{\sqrt{2}}{n}$ is irrational, and

$$a < a + \frac{\sqrt{2}}{n} < a + \frac{\sqrt{2}}{\sqrt{2}/(b-a)} = b.$$

Case 2: If a is irrational, then $a + \frac{1}{n}$ is irrational, and

$$a < a + \frac{1}{n} < a + \frac{\sqrt{2}}{n} < a + \frac{\sqrt{2}}{\sqrt{2}/(b-a)} = b.$$

□

2 Decimals

Lecture 7

Proposition 6. *Let x be a real number.*

(1) *If $x \neq 1$, then*

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

(2) *If $|x| < 1$, then*

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}.$$

The decimal expression $a_0.a_1a_2a_3\cdots$ (in base 10), where a_0 is an integer, and $a_1, a_2, \ldots$ are numbers from 0 to 9, is the real number

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots .$$

Proposition 7. *Every real number x has a decimal expression*

$$x = a_0.a_1a_2a_3\cdots .$$

Proof. The real number x must lie between two consecutive integers, say, a_0 and $a_0 + 1$, so that

$$a_0 \leq x < a_0 + 1.$$

Now we divide the interval $[a_0, a_0 + 1]$ into 10 equal small intervals. Clearly, x lies in one of these small intervals. So we can find an integer a_1 between 0 and 9 inclusive so that

$$a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1 + 1}{10}.$$

Similarly, we divide the interval $[a_0 + \frac{a_1}{10}, a_0 + \frac{a_1+1}{10}]$ into 10 equal smaller intervals; then x lies in one of these smaller intervals; and we can find an integer a_2 between 0 and 9 inclusive so that

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} \leq x < a_0 + \frac{a_1}{10} + \frac{a_2 + 1}{10^2}.$$

Continuing this procedure, we obtain a sequence

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n},$$

which gets as close as we like to x when n is large enough. This is what we mean the decimal expression $a_0.a_1a_2a_3\cdots$ of x. $\square$

Example 1. Show that $\sqrt{3} \approx 1.732$.

Proof. Let $x = \sqrt{3}$. Since $x^2 = 3$, then $1^2 = 1 < x^2 < 4 = 2^2$, so $1 < x < 2$, thus $a_0 = 1$. Next, $(1.7)^2 = 2.89 < x^2 < 3.24 = (1.8)^2$, then $1.7 < x < 1.8$, we have $a_1 = 7$. Similarly, $(1.73)^2 = 2.9929 < x^2 < 3.0276 = (1.74)^2$, then $1.73 < x < 1.74$, we have $a_2 = 3$. Since $(1.731)^2 = 2.996361 < x^2 < 3.003288 = (1.732)^2$, then $1.731 < x < 1.732$, and $a_3 = 1$. Note that $(1.7319)^2 = 2.99947761 < x^2 < 3.003288 = (1.732)^2$. We have $1.7319 < x < 1.732$. Hence $x \approx 1.732$. $\square$

Question 1. Can the same real number have two different decimal expressions? If Yes, which decimal expressions represent the same real number?

$$\begin{aligned} 0.999\cdots &= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots \\ &= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) \\ &= \frac{9}{10} \cdot \frac{1}{1 - 1/10} = 1. \end{aligned}$$

Thus

$$1 = 1.000\cdots = 0.999\cdots.$$

Similarly,

$$\frac{369}{1000} = 0.368999\cdots = 0.369000\cdots.$$

Proposition 8. *If a real number x is expressed (in base 10) in two different expressions:*

$$a_0.a_1a_2a_3\cdots \quad \textit{and} \quad b_0.b_1b_2b_3\cdots,$$

then one of these expressions ends in $999\cdots$ and the other ends in $000\cdots$.

Proof. Let k be the left most position where $a_k \neq b_k$, $k \in \mathbb{Z}_{\geq 0}$. Then $a_0 = b_0$, $a_1 = b_1$, $\ldots$, $a_{k-1} = b_{k-1}$. Without loss of generality, we may assume $a_k > b_k$. Thus $a_k \geq b_k + 1$. Since $x = a_0.a_1a_2\cdots = b_0.b_1b_2\cdots$, we have

$$a_0.a_1a_2\cdots a_k00\cdots \leq a_0.a_1a_2\cdots = x = b_0.b_1b_2\cdots \leq b_0.b_1b_2\cdots b_k999\cdots.$$

That is,

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_{k-1}}{10^{k-1}} + \frac{a_k}{10^k} \leq x \leq b_0 + \frac{b_1}{10} + \cdots + \frac{b_{k-1}}{10^{k-1}} + \frac{b_k}{10^k} + 9\left(\frac{1}{10^{k+1}} + \frac{1}{10^{k+2}} + \cdots\right).$$

It follows that

$$a_k \leq b_k + 9\left(\frac{1}{10^1} + \frac{1}{10^2} + \cdots\right) = b_k + 1 \leq a_k.$$

Hence $a_k = b_k + 1$, and

$$x = a_0.a_1a_2\ldots a_k000\cdots = b_0.b_1b_2\ldots b_k999\cdots.$$

We then have that $a_0.a_1a_2\cdots$ ends with $000\cdots$ and $b_0.b_1b_2\cdots$ ends with $999\cdots$. $\square$

For rational numbers $\frac{18}{7}$, and 821, we have

$$\frac{18}{7} = 2.571428571428571428\cdots = 2.\overline{571428},$$

$$\frac{8}{21} = 0.380952380952380952\cdots = 0.\overline{380952}.$$

Proposition 9. *A real number x is rational if and only if its decimal expression is periodic.*

Proof. Let $x = \frac{m}{n}$ be a rational number in reduced form, where n is a positive integer. Do the following division to have quotients and remainders:

$$\left.\begin{array}{l}
\begin{array}{rcll}
m & = & q_0 n + r_0, & 0 \leq r_0 < n, \\
10r_0 & = & q_1 n + r_1, & 0 \leq r_1 < n, \\
10r_1 & = & q_2 n + r_2, & 0 \leq r_2 < n, \\
\cdots & \cdots & \cdots\cdots\cdots & \cdots\cdots\cdots
\end{array} \\
\left.\begin{array}{rcll}
10r_{k-1} & = & q_k n + r_k, & 0 \leq r_k < n, \\
10r_k & = & q_{k+1} n + r_{k+1}, & 0 \leq r_{k+1} < n, \\
\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots
\end{array}\right\} l
\end{array}\right\} \text{steps } \leq n-1$$

$$\left.\begin{array}{rcll}
10r_{k+l-1} & = & q_{k+l} n + r_{k+l}, & r_{k+l} = r_k, \\
10r_{k+l} & = & q_{k+l+1} n + r_{k+l+1}, & r_{k+1+1} = r_{k+1}, \\
\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots
\end{array}\right\} l$$

$$\vdots$$

Since the remainders dividing by n can be only $0, 1, 2, \ldots, n-1$, the remainders must repeat periodically. Thus $q_{k+l+1} = q_{k+1}$, $q_{k+l+2} = q_{k+2}$, $\ldots$; that is, $q_{a+il} = q_a$ for $a \geq k+1$.

$$\begin{aligned}
\frac{m}{n} &= q_0.q_1q_2\cdots q_k\,\underbrace{q_{k+1}q_{k+2}\cdots q_{k+l}}_{l}\,\underbrace{q_{k+l+1}q_{k+l+2}\cdots q_{k+2l}}_{l}\cdots \\
&= q_0.q_1q_2\cdots q_k\,\underbrace{q_{k+1}q_{k+2}\cdots q_{k+l}}_{l}\,\underbrace{q_{k+1}q_{k+2}\cdots q_{k+l}}_{l}\cdots \\
&= q_0.q_1q_2\cdots q_k\overline{q_{k+1}q_{k+2}\cdots q_{k+l}}.
\end{aligned}$$

It is clear that $1 \le l \le n$. Moreover, if $n \ge 2$, then $l \le n-1$. In fact, if one of the remainders r_i is zero then all the following remainders are zero; so $l=1$. Otherwise, all remainder r_i are nonzero. Of course, $l \le n-1$.

$\square$

Conversely, given a number $x = a_0.a_1a_2\dots a_k\overline{q_1q_2\dots q_l}$ having periodic decimal expression. Then

$$x = a_0 + \frac{a_1}{10} + \cdots + \frac{a_k}{10^k} + r,$$

where

$$\begin{aligned}
r &= \frac{1}{10^k}\left(\frac{q_1}{10} + \cdots + \frac{q_l}{10^l}\right) + \frac{1}{10^{k+l}}\left(\frac{q_1}{10} + \cdots + \frac{q_l}{10^l}\right) + \frac{1}{10^{k+2l}}\left(\frac{q_1}{10} + \cdots + \frac{q_l}{10^l}\right) + \cdots \\
&= \frac{1}{10^k}\left(\frac{q_1}{10} + \cdots + \frac{q_l}{10^l}\right)\left(1 + \frac{1}{10^l} + \frac{1}{10^{2l}} + \cdots\right) \\
&= \frac{1}{10^k}\left(\frac{q_1}{10} + \cdots + \frac{q_l}{10^l}\right)\frac{10^l}{10^l - 1}.
\end{aligned}$$

Example 2.

$$\begin{aligned}
1.6\overline{18} &= 1 + \frac{6}{10} + \frac{1}{10^2}\left(1 + \frac{8}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{1}{10^4} + \frac{8}{10^5} + \cdots\right) \\
&= 1 + \frac{6}{10} + \frac{1}{10^2}\cdot\frac{10^2}{99} + \frac{8}{10^3}\cdot\frac{10^2}{99} \\
&= 1 + \frac{6}{10} + \frac{1}{99} + \frac{8}{990} = \frac{1602}{990} = \frac{89}{55}.
\end{aligned}$$

Lecture 9

3 Inequalities

An **inequality** is a statement about real numbers involving one of the symbols $>$, $\ge$, $<$, or $\le$. We start with the following rules about inequalities. The following rules of real numbers are motivated by the properties of the real axis – the set of real numbers.

Definition 10. Rules of Inequalities

1. For each $x \in \mathbb{R}$, then either $x < 0$ or $x = 0$ or $x > 0$, and just one of these three is true.
2. If $x < y$, $y < z$, then $x < z$.
3. If $x < y$ and $c \in \mathbb{R}$, then $x + c < y + c$.
4. If $x > 0, y > 0$, then $xy > 0$.
5. If $x < y$, then $-x > -y$.

For two real numbers x, y, we use $x \le y$ to denote either $x < y$ or $x = y$. Analogously, $x \ge y$ denotes either $x > y$ or $x = y$.

Example 3. 1. If $x < 0$, then $-x > 0$. If $x \ge 0$, then $-x \le 0$.

2. If $x \ne 0$, then $x < 0$ or $x > 0$, and $x^2 > 0$.

3. If $x < y$ and $u > 0$, then $ux < uy$.

   Proof. Since $x < y$, then $x - x < y - x$, that is, $0 < y - x$. Thus $u \cdot 0 < u(y - x)$, that is, $0 < uy - ux$. Hence, $0 + ux < uy - ux + ux$, that is, $ux < uy$. □

4. If $x > 0$, then $\frac{1}{x} > 0$.

   Proof. Suppose $\frac{1}{x} < 0$. Then $\frac{-1}{x} > 0$. Thus $x \cdot \frac{-1}{x} > 0$, that is, $-1 > 0$, this is a contradiction. So $\frac{1}{x} \geq 0$. Since $\frac{1}{x} \neq 0$, we conclude that $\frac{1}{x} > 0$. □

Example 4. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ be nonzero. Assume k of them are negative and the rest are positive. Then

$$x_1 x_2 \cdots x_n = \begin{cases} > 0 & \text{if } k \text{ is even,} \\ < 0 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Without loss of generality we may assume that $x_1, \ldots, x_k$ are negative and $x_{k+1}, \ldots, x_n$ are positive. Then $-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_n$ are all positive. Thus

$$(-1)^k x_1 x_2 \cdots x_n = (-x_1) \cdots (-x_k) x_{k+1} \cdots x_n > 0.$$

If k is even, the above inequality means that $x_1 x_2 \cdots x_n > 0$. If k is odd, $-x_1 x_2 \cdots x_n > 0$; thus $x_1 x_2 \cdots x_n < 0$. □

Example 5. For which values of x is $x < \frac{3}{x+2}$?
Answer. We cannot multiply $x + 2$ to both side as $x + 2$ may not be all positive or all negative. instead, we do

$$x - \frac{3}{x+2} - x < 0 \quad \Longleftrightarrow \quad \frac{x(x+2) - 3}{x+2} = \frac{(x+3)(x-1)}{x+2} < 0.$$

To have the product of the three terms $x + 3$, $x - 1$, $\frac{1}{x+2}$ to be negative, we have two situations: (i) one of the three is negative and the other two are positive; (ii) all three are negative. In the former case, we have

(1) $x + 3 < 0$, $x - 1 > 0$, and $x + 2 > 0$, that is, $x < -3$, $x > -1$, $x > -2$. No such value x.
(2) $x + 3 > 0$, $x - 1 < 0$, $x + 2 > 0$, that is, $x > -3$, $x < 1$, $x > -2$. Then $-2 < x < 1$.
(3) $x + 3 > 0$, $x - 1 > 0$, $x + 2 < 0$, that is, $x > -3$, $x > 1$, $x < -2$. No such value x.

In the latter case,

$$x + 3 < 0,\ x - 1 < 0,\ x + 2 < 0 \quad \Longleftrightarrow \quad x < -3,\ x < 1,\ x < -2 \quad \Longleftrightarrow \quad x < -3.$$

So our answer is $x < -3$ or $-2 < x < 1$, that is $x \in (-\infty, 3) \cup (-2, 1)$.

Example 6. Show that for all real numbers x we have $x^2 + 3x + 3 > 0$.

Proof. Note that $x^2 + 3x + 3 = \left(x + \frac{3}{2}\right)^2 + \frac{3}{4}$. Since $\left(x + \frac{3}{2}\right)^2 \geq 0$ and $\frac{3}{4} > 0$, then $\left(x + \frac{3}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0$. So $x^2 + 3x + 3 > 0$. □

The **modulus** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Example 7. For a positive real number r, $|x| < r$ means $-r < x < r$; $|x| \leq r$ means $-r \leq x \leq r$.

For $a, r \in \mathbb{R}$ with $r > 0$, we have

$$|x - a| < r \;\Leftrightarrow\; a - r < x < a + r; \quad |x - a| \leq r \;\Leftrightarrow\; a - r \leq x \leq a + r.$$

4 Rational Powers

Theorem 11 (Definition)**.** *Let n be a positive integer. For each positive real number b, there exists a unique positive real number x such that $x^n = b$. We write the number x in terms of b as*

$$x = b^{\frac{1}{n}}.$$

Let b be a positive real number $b > 0$. For rational numbers $\frac{m}{n} \in \mathbb{Q}$ with $n > 0$ and $m \in \mathbb{Z}$, we define the **rational power** (also known as **fractional power**) of b to $\frac{m}{n}$ as the positive real number

$$b^{\frac{m}{n}} := (b^{\frac{1}{n}})^m.$$

We need to show that $b^{\frac{m}{n}}$ is well-defined when $\frac{m}{n}$ is not in reduced form. Let $\frac{m}{n}$ be in reduced form and consider $\frac{mk}{nk}$ with $k \in \mathbb{Z}_+$. For the positive integer $b^{\frac{1}{n}}$, there exists a unique positive integer a such that $a^k = b^{\frac{1}{n}}$, that is, $a = (b^{\frac{1}{n}})^{\frac{1}{k}}$. Let us write $y = b^{\frac{1}{nk}}$, that is, $y^{nk} = b$. Thus

$$a^{nk} = \left[(b^{\frac{1}{n}})^{\frac{1}{k}}\right]^{nk} = \left[\left((b^{\frac{1}{n}})^{\frac{1}{k}}\right)^k\right]^n = (b^{\frac{1}{n}})^n = b.$$

This means that $y = a$. Therefore

$$b^{\frac{mk}{nk}} = (b^{\frac{1}{nk}})^{mk} = y^{mk} = a^{mk} = (a^k)^m = \left(b^{\frac{1}{n}}\right)^m = b^{\frac{m}{n}}.$$

For instance, $7^{-\frac{13}{5}} = (7^{\frac{1}{5}})^{-13} = \frac{1}{(\sqrt[5]{7})^{13}}$.

Proposition 12 (Power Rules)**.** *Let $x, y \in \mathbb{R}_+$ and $p, q \in \mathbb{Q}$. Then*

(a) $x^p x^q = x^{p+q}$.

(b) $(x^p)^q = x^{pq}$.

(c) $(xy)^p = x^p y^p$.

Proof. (a) We first assume that $p, q \in \mathbb{Z}$. It is trivial when $p = 0$ or $q = 0$. If $p, q > 0$, then

$$x^p x^q = \underbrace{x \cdots x}_{p}\underbrace{x \cdots x}_{q} = \underbrace{x \cdots x}_{p+q} = x^{p+q}.$$

If $p > 0, q < 0$, then

$$x^p x^q = \underbrace{x \cdots x}_{p} / \underbrace{x \cdots x}_{-q} = x^{p-(-q)} = x^{p+q}.$$

It is similar when $p < 0, q > 0$ and when $p, q < 0$. Now Let $p = \frac{m}{n}$, $q = \frac{h}{k}$. Then

$$x^p x^q = x^{\frac{m}{n}} x^{\frac{h}{k}} = x^{\frac{mk}{nk}} x^{\frac{nh}{nk}} = \left(x^{\frac{1}{nk}}\right)^{mk}\left(x^{\frac{1}{nk}}\right)^{nh} = \left(x^{\frac{1}{nk}}\right)^{mk+nh} = x^{\frac{mk+nh}{nk}} = x^{p+q}.$$

(b) We first establish the rule for $p, q \in \mathbb{Z}$. It is obviously true when $p = 0$ or $q = 0$. If $p > 0, q > 0$, it is trivial. If $p > 0, q < 0$, then $(x^p)^q = \frac{1}{(x^p)^{-q}} = \frac{1}{x^{-pq}} = x^{pq}$. If $p < 0, q > 0$, then $(x^p)^q = \left(\frac{1}{x^{-p}}\right)^q = \frac{1}{x^{-pq}} = x^{pq}$. If $p, q < 0$, then $(x^P)^q = 1/(\frac{1}{x^{-p}})^{-q} = 1/\frac{1}{(x^{-p})^{-q}} = 1/\frac{1}{x^{pq}} = x^{pq}$.

Let $p = \frac{m}{n}$, $q = \frac{h}{k}$ with $m, n, h, k \in \mathbb{Z}$. It follows from Theorem 11 that there exists a positive real number a such that $x = a^{nk}$, that is, $a = x^{\frac{1}{nk}}$, and there exists a positive real number b such that $a^k = b^{\frac{1}{n}}$. Then $a^{nk} = (a^k)^n = b$. So $b = x$, that is, $a^k = x^{\frac{1}{n}}$. Thus $(x^{\frac{1}{nk}})^k = x^{\frac{1}{n}}$. Therefore

$$\begin{aligned}(x^p)^q &= (x^{\frac{m}{n}})^{\frac{h}{k}} = [(x^{\frac{mk}{nk}})^{\frac{1}{k}}]^h = \left[\left((x^{\frac{1}{nk}})^{mk}\right)^{\frac{1}{k}}\right]^h = \left[\left(((x^{\frac{1}{nk}})^m)^k\right)^{\frac{1}{k}}\right]^h \\ &= \left[\left(x^{\frac{1}{nk}}\right)^m\right]^h = \left(x^{\frac{1}{nk}}\right)^{mh} = x^{\frac{mh}{nk}} = x^{pq}.\end{aligned}$$

(c) It is trivial to establish the rule for $p \in \mathbb{Z}$. Let $p = \frac{m}{n}$. Then

$$x^p y^p = (x^{\frac{1}{n}})^m (y^{\frac{1}{n}})^m = (x^{\frac{1}{n}} y^{\frac{1}{n}})^m = \left[\left((x^{\frac{1}{n}} y^{\frac{1}{n}})^n\right)^{\frac{1}{n}}\right]^m = \left[(x^{\frac{1}{n}})^n (y^{\frac{1}{n}})^n\right]^{\frac{m}{n}} = (xy)^p.$$

$\square$

5 Complex Numbers

A **complex number** z is a combination of real numbers written in the form

$$z = a + bi,$$

where the addition and multiplication are the same as the operations of algebraic terms, with an additional rule $i^2 = -1$; a is called the **real part** of z, and b the **imaginary part**, and we write

$$a = \mathrm{Re}(z), \quad b = \mathrm{Im}(z).$$

We denote by $\mathbb{C}$ the set of all complex numbers.

For any real number a, it is automatically a complex number with $\mathrm{Im}(a) = 0$; we write a instead of $a + 0i$ without mentioning the zero imaginary part. The real number 0 is still the zero in complex numbers as $0 + z = z$ for any complex number z; the real number 1 is still the unit for complex number as $1z = z$ for any complex number z. For each complex number $z = a + bi$, the complex number $\bar{z} = a - bi$ is called the **conjugate** of z, and $|z| = \sqrt{a^2 + b^2}$ is called the **modulus** of z; $|z|^2 = z\bar{z} = \bar{z}z = a^2 + b^2$.

The **minus** of z is defined as a complex number w such that $z + w = 0$, and it is denoted by $-z$. If $z = a + bi$, then $-z = -a - bi$. The **subtract** of a complex number w from a complex number z is defined as

$$z - w = z + (-w).$$

Similarly, the **inverse** of a complex number $z (\neq 0)$ is defined as a complex number w such that $zw = 1$; the inverse of z is denoted by $\frac{1}{z}$ or z^{-1}. Since $0w = 0$ for any $w \in \mathbb{C}$, there is no (complex) inverse for 0. If $z = a + bi \neq 0$, then $zz^{-1} = 1$; multiplying both sides by $\bar{z} = a - bi$, we have

$$\bar{z}zz^{-1} = \bar{z}, \quad \text{i.e.} \quad |z|^2\bar{z} = (a^2 + b^2)z^{-1} = \bar{z}.$$

Hence

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

Thus for complex numbers w and z with $z \neq 0$, the division $\frac{w}{z}$ is defined as

$$\frac{w}{z} = wz^{-1}.$$

If $z = a + bi$ and $w = c + di$, then

$$\frac{w}{z} = \frac{w\bar{z}}{|z|^2} = \frac{(c + di)(a - bi)}{a^2 + b^2} = \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}i.$$

6 De Moivre's Rule

For complex number $z = a + bi$, let $r = \sqrt{a^2 + b^2} = |z|$. Then $a = r\cos\theta$ and $b = r\sin\theta$, and z can be written as

$$z = r(\cos\theta + i\sin\theta).$$

Theorem 13. *Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_1(\cos\theta_2 + i\sin\theta_2)$. Then*

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Proof. Recall the trigonometric formulas:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \quad \sin(\theta_1 + \theta_2) = \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.$$

Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2[\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2] \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

□

Lecture 10

Corollary 14. *Let $z = r(\cos\theta + i\sin\theta)$. Then for any integer n,*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof. For positive integer n it is easy to apply the De Moivre's rule. Note that

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{r}(\cos\theta - i\sin\theta) = r^{-1}[\cos(-\theta) + i\sin(-\theta)] = r_1(\cos\theta_1 + i\sin\theta_1),$$

where $r_1 = r^{-1}$ and $\theta_1 = -\theta$. Then for positive integer n,

$$z^{-n} = r_1^n(\cos n\theta_1 + i\sin n\theta_1) = r^{-n}\Big(\cos(-n\theta) + i\sin(-n\theta)\Big).$$

□

Definition 15. For any angle θ the complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$, i.e.,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Recall the trigonometric functions $\cos\theta$ and $\sin\theta$ are defined by

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}.$$

where $x^2 + y^2 = r^2$.

Theorem 16.

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

Example 8. Computer $(-1 + \sqrt{3}i)^{20}$.

Let $\alpha = -1 + \sqrt{3}i$. Then $\alpha = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$. Thus

$$\alpha^{20} = 2^{20}\left(\cos\frac{40\pi}{3} + i\sin\frac{40\pi}{3}\right) = 2^{20}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2^{19}(-1-\sqrt{3}i).$$

Example 9. Deriving trigonometric formulas. Consider $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$. Let $a = \cos\theta$, $b = \sin\theta$. Then

$$\begin{aligned}(a+bi)^3 &= (a^2 - b^2 + 2abi)(a+bi)\\ &= (a^2-b^2)a - 2ab^2 + (2a^2b + a^2b - b^3)i\\ &= a^3 - 3ab^2 + (3a^2b - b^3)i.\end{aligned}$$

Thus

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta.$$

Similarly,

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Proposition 17. (a) *If $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.*

(b) *Let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$. Then $z_1 = z_2$ if, and only if, $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ for some $k \in \mathbb{Z}$.*

Proof. (a) is obvious. (b) If $z_1 = z_2$, then $r_1 = r_2$, and $1 = z_1/z_2 = e^{i(\theta_1-\theta_2)}$. Hence $\theta_1 - \theta_2 = 2k\pi$ for some $k \in \mathbb{Z}$. The other part is obvious. □

7 Roots of unity

Definition 18. For any positive integer n, let $w = e^{\frac{2\pi i}{n}}$; the nth roots of unity are the complex numbers

$$1, w, w^2, \ldots, w^{n-1}.$$

They are evenly distributed on the unit circle.

Example 10. For $n = 2$, they are $1, -1$. For $n = 4$, they are $1, i, -1, -i$. For $n = 3$, they are

$$1, \quad e^{\frac{2\pi i}{3}}, \quad e^{\frac{4\pi i}{3}}.$$

Theorem 19. *For the nth root of unity $w = e^{\frac{2\pi i}{n}}$ with $n \geq 2$,*

$$1 + w + w^2 + \cdots + w^{n-1} = 0.$$

Proof. Since $w^n = 1$ and $1 - w \neq 0$, then

$$(1-w)(1 + w + \cdots + w^{n-1}) = 1 - w^n = 0.$$

Hence $1 + w + \cdots + w^{n-1}$ must be zero. □

When a complex number $z = a + bi$ is interpreted as an vector or force from the origin $(0,0)$ to the position (a,b), the physical meaning of the above identity means that the sum effect of the forces $1, w, w^2, \ldots, w^{n-1}$ cancels each other at the origin.

Lecture 11

8 Cubic Equations (optional)

The general cubic equation may be written as

$$x^3 + ax^2 + bx + c = 0. \tag{1}$$

Let $x = x - \frac{a}{3}$. Then $x^3 = (y - a/3)^3 = y^3 - ay^2 + (a^2/3)y - a^3/27$, $y^2 = x^2 - (2a/3)y + a^2/9$. Substitute $x = y - a/3$ into (1); the equation becomes the form

$$y^3 + 3hy + k = 0. \tag{2}$$

Let $y = u + v$. Then

$$y^3 = u^3 + v^3 + 3u^2v + 3uv^2 = u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvy.$$

This means that the equation of the form $y^3 - 3uvy - (u^3 + v^3) = 0$ readily has a solution $y = u + v$. So we set

$$h = -uv, \quad k = -(u^3 + v^3).$$

Since $v = -h/u$, then $v^3 = -h^3/u^3$. Thus $k = -(u^3 - h^3/u^3)$ becomes

$$u^6 + ku^3 - h^3 = 0,$$

which is a quadratic equation in u^3. Then u^3 as

$$u^3 = \frac{-k + \sqrt{k^2 + 4h^3}}{2}.$$

Thus

$$v^3 = -k - u^3 = \frac{-k - \sqrt{k^2 + 4h^3}}{2}.$$

Therefore we obtain a solution

$$y = u + v = \sqrt[3]{\frac{-k + \sqrt{k^2 + 4h^3}}{2}} + \sqrt[3]{\frac{-k - \sqrt{k^2 + 4h^3}}{2}}.$$

There are three cubic roots for $u^3 = \frac{-k+\sqrt{k^2+4h^3}}{2}$ and also three cubic roots for $v^3 = \frac{-k-\sqrt{k^2+4h^3}}{2}$. So theoretically there are nine possible values to be the solutions; but there are only three solutions, some of them are the same.

Let u be a cubic root of $\frac{-k+\sqrt{k^2+4h^3}}{2}$, and let $\omega = e^{2\pi i/3}$. Then the other two cubic roots are $u\omega, u\omega^2$. Therefore the solutions for (2) are given by

$$u - \frac{h}{u}, \qquad u\omega - \frac{h\omega^2}{u}, \qquad u\omega^2 - \frac{h\omega}{u}.$$

Example 11. Consider the equation

$$x^3 - 3x + 2 = 0.$$

Since $h = -1$, $k = 2$, we have

$$u^3 = \frac{-k + \sqrt{k^2 + 4h^3}}{2} = -1.$$

So we have $u = -1$, thus the three solutions are given by

$$u - \frac{h}{u} = -2,$$

$$u\omega - \frac{h\omega^2}{u} = -\omega - \omega^2 = 1 - (1 + \omega + \omega^2) = 1,$$

$$u\omega^2 - \frac{h\omega}{u} = -\omega^2 - \omega = 1.$$

We may also solve the problem directly by the factorization $(x-1)(x-1)(x+2) = 0$.

Example 12. Consider the equation

$$x^3 - 6x - 6 = 0.$$

We have $h = -2$ and $k = -6$. Thus

$$u^3 = \frac{-k + \sqrt{k^2 + 4h^3}}{2} = 4.$$

So $u = \sqrt[3]{4}$. Thus

$$x_1 = u - \frac{h}{u} = 4^{1/3} + 2/4^{1/3} = 2^{2/3} + 2^{1/3},$$

$$x_2 = u\omega - \frac{h\omega^2}{u} = (2^{1/3} + 2^{2/3})\omega + (2^{-1/3} + 2^{-2/3})^{-1}\omega^2,$$

$$x_3 = u\omega^2 - \frac{h\omega}{u} = (2^{-1/3} + 2^{-2/3})^{-1}\omega + (2^{1/3} + 2^{2/3})\omega^2.$$

9 Fundamental Theorem of Algebra

Theorem 20. *Every polynomial equation of degree at leat* 1 *has a root in* $\mathbb{C}$.

Theorem 21. *Every polynomial of degree n factories as a product of linear polynomials, and has exactly n roots (counted with multiplicity) in* $\mathbb{C}$.

Proposition 22. *Let $\alpha_1, \ldots, \alpha_n$ be the roots of the equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Then

$$s_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_n = -a_{n-1}$$

$$s_2 = \sum_{i<j} \alpha_i \alpha_j = a_{n-2},$$

$$s_3 = \sum_{i<j<k} \alpha_i \alpha_j \alpha_k = -a_{n-3},$$

$$\cdots,$$

$$s_k = \sum_{i_1<i_2<\cdots<i_k} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k} = (-1)^k a_{n-k},$$

$$\cdots,$$

$$s_n = \alpha_1\alpha_2\cdots\alpha_n = (-1)^n a_0.$$

Example 13. Find a cubic equation with roots $2+i$, $2-i$, and 3.

$$s_1 = \alpha_1 + \alpha_2 + \alpha_3 = 7,$$

$$s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = 17,$$

$$s_3 = \alpha_1\alpha_2\alpha_3 = 15.$$

$$(x-3)(x^2-4x+5) = x^3 - 7x^2 + 17x - 15 = 0.$$

Example 14. Let α and β be roots of equation $x^2 - 5x + 9 = 0$. Find a quadratic equation with roots α^2 and β^2.

The quadratic equation is of the form

$$x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0.$$

Since $\alpha + \beta = 5$ and $\alpha\beta = 9$, we have $5^2 = (\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = \alpha^2 + \beta^2 + 18$. Then $\alpha^2 + \beta^2 = 7$, $\alpha^2\beta^2 = 81$. Thus

$$x^2 - 7x + 81 = 0.$$